

Chiral Quantization of the WZW $SU(n)$ Model

L. Caneschi

Dept. of Physics, Univ. of Ferrara and INFN Ferrara, Italy

M. Lysiansky *

Dept. of Physics of Complex Systems, Weizmann Institute of Science, Rehovot, Israel

Abstract

We quantize the $SU(n)$ Wess-Zumino-Witten model in terms of left and right chiral variables choosing an appropriate gauge and we compare our results with the results that have been previously obtained in the algebraic treatment of the problem. The algebra of the chiral vertex operators in the fundamental representation is verified by solving appropriate Knizhnik-Zamolodchikov equations.

*E-mail: famisha@wishful.weizmann.ac.il

1. INTRODUCTION

Rational conformal field theories are well understood from the algebraic point of view [1] [2] [3]. The structure is determined by the existence of commuting chiral algebras $\mathcal{A} \times \bar{\mathcal{A}}$. The Hilbert space consists of the direct product of infinite dimensional representations of these algebras, while the chiral vertex operators are intertwining operators between these representations. The way in which the left and right representations are paired is determined by modular invariance. The chiral operators are coupled together to produce "physical operators" which are local with respect to each other.

It is an interesting question to study in which way the aforementioned structure is encoded into the lagrangian describing the conformal theory, when such a description exists. This problem was studied for the Wess-Zumino-Witten (WZW) class of conformal theories where the chiral algebra is the Kac-Moody one. These theories are described by the WZW lagrangian [4]. The basic idea [5] [6] [7] in establishing the connection between the lagrangian and the algebraic descriptions is to quantize the theory in a parameterization of the phase space which is chiral, i. e. the traditional field and conjugate momentum are replaced by group valued functions representing the left and right moving classical solutions.

This approach was further developed in [8]. In the present work we continue the study of this problem for the SU (n) WZW theory. The main results obtained are :

- a) A complete description of the Hilbert space required by the chiral quantization which is shown to be identical to the one required by the algebraic treatment.
- b) The exact correspondence between the chiral vertex operators of the algebraic treatment and the chiral group elements of the lagrangian one.
- c) The correspondence between the mutually local physical operators and appropriate combinations of chiral group elements.

In solving the problems mentioned above we found very useful the chiral quantization proposed in [5] [6] [8] of a "toy model", the motion on a group manifold.

In Section 2 we solve completely in a chiral quantization the motion on a SU (n) group

manifold. A technical problem which appears in the chiral quantization is the appropriate choice of a gauge. We compare the chiral solution and the conventional one and we prove explicitly that our choice leads to the correct mapping between chiral and standard operators. Since the zero modes of the WZW model are described by the chiral lagrangian of the "toy model" this indicates that an analogous gauge choice is justified also for the WZW model.

In Section 3 we rederive the lagrangian representing the WZW model in chiral variables. We give an explicit description of the Hilbert space and of the matrix elements of the chiral group elements and using the Knizhnik-Zamolodchikov equations we prove that the chiral group elements satisfy the correct commutation relations.

In Section 4 we summarize the results and list the problems that are still unsolved.

2. CHIRAL QUANTIZATION OF THE MOTION ON A GROUP MANIFOLD

A. The standard treatment

Let us consider the free motion on the compact simple Lie group G . We define $t^a = \{H^i, E^\alpha\}$ for $i = 1, \dots, r$, where r is the rank of G and α are its roots to be a basis of the generators for the corresponding Lie algebra \mathcal{G} . We choose the normalization such that $tr(t^a t^b) = \delta_{ab}$. The action for the motion of a "free particle" on the group manifold is :

$$S = -\frac{1}{2} \int tr(g^{-1} \dot{g} g^{-1} \dot{g}) dt, \quad g \in G \quad (1)$$

This action is invariant under transformations belonging to the group $G \times G$, corresponding to the multiplication of g with arbitrary group elements from the left and right respectively.

The Hilbert space is spanned by the group elements. Alternatively [9] one can use the dual basis of matrix elements of all the unitary irreducible representations labeled by the highest weight of the representation λ and two set of "magnetic" quantum numbers which we denote collectively by μ_l and μ_r respectively.

The hamiltonian is diagonal in the dual basis, the eigenvalue E_λ being the second order Casimir of the group which depends just on the highest weight :

$$E_\lambda = \lambda(\lambda + \varrho) \quad (2)$$

where ϱ is the half sum of the positive roots. Since the hamiltonian is diagonalized it is easy to calculate the propagator $K(g(t), g(0))$ from an initial state characterized by an eigenvalue of g , $g(0)$ to a final state $g(t)$:

$$K(g(t), g(0)) = \sum_{\lambda, \mu_l, \mu_r} D_{\mu_l, \mu_r}^{\lambda*}(g(t)) D_{\mu_l, \mu_r}^{\lambda}(g(0)) \exp(itE_\lambda) \quad (3)$$

In the next subsection we quantize this simple model in the chiral quantization and compare the results with the above formulae.

B. Classical analysis in chiral variables

Let us consider the equations of motion corresponding to the action (1) :

$$\frac{d}{dt}(g^{-1}\dot{g}) = \frac{d}{dt}(\dot{g}g^{-1}) = 0 \quad (4)$$

We can write the solution in the form :

$$g(t) = ue^{i\vec{p}\vec{H}t}v, \quad u, v \in G \quad (5)$$

There is an ambiguity in the parametrization since replacing

$$u \rightarrow uh, \quad v \rightarrow h^{-1}v \quad (6)$$

where h is an arbitrary element from the maximal torus T leads to the same solution.

The equations of motion can also be written in the Hamiltonian form :

$$\omega = g^{-1}\dot{g}, \quad \omega \in \mathcal{G} \quad (7)$$

$$\dot{\omega} = 0$$

These equations can be obtained from the action :

$$S = - \int \text{tr}(\omega g^{-1}\dot{g} - \frac{\omega^2}{2})dt \quad (8)$$

The action (8) is first order and ω , g are the phase-space variables.

To factorize the left and right theories we make the change of variable from ω and g to u and v . The phase space is the space of all classical solutions. In the standard treatment g , ω describe this space by representing the initial conditions on a classical trajectory $g = g(t)$:

$$\omega = g^{-1}\dot{g}|_{t=0} \quad (9)$$

$$g = g(t)|_{t=0}$$

Using the chiral parametrization of the solution (9) and (9) we obtain :

$$\omega = iv^{-1}\vec{p}\vec{H}v \quad (10)$$

$$g = uv$$

Here \vec{p} is a periodic variable, the periodicity being defined by the root lattice. In the new variables the action is :

$$S = -i \int tr(\vec{p}\vec{H}(u^{-1}\dot{u} + \dot{v}v^{-1} - i\frac{\vec{p}\vec{H}}{2}))dt \quad (11)$$

The ambiguity in the parametrization mentioned above (6) manifests itself as a gauge invariance of the action (11) under the transformation (6). As a consequence the symplectic Ω form derived from the action (11) :

$$\Omega = -itr(d\vec{p}\vec{H}(u^{-1}du + dvv^{-1}) - \vec{p}\vec{H}(u^{-1}duu^{-1}du + dvv^{-1}dvv^{-1})) \quad (12)$$

is degenerate and cannot be inverted.

To avoid this problem we fix the gauge in a convenient way introducing new variables \vec{p}_l and \vec{p}_r in terms of which the action takes the form :

$$S = -i \int tr(\vec{H}\vec{p}_l u^{-1}\dot{u} + \vec{H}\vec{p}_r \dot{v}v^{-1} - \frac{i}{8}((\vec{p}_l + \vec{p}_r)\vec{H})^2)dt \quad (13)$$

The relevant symplectic form now is non-degenerate :

$$\tilde{\Omega} = -itr(d\vec{p}_l \vec{H} u^{-1} du - \vec{p}_l \vec{H} u^{-1} u u^{-1} u + d\vec{p}_r \vec{H} dv v^{-1} + \vec{p}_r \vec{H} dv v^{-1} dv v^{-1}) \quad (14)$$

The action (13) is equivalent to (11) provided that the constraint :

$$\vec{\phi} = \vec{p}_l - \vec{p}_r = 0 \quad (15)$$

is fulfilled. Equation (15) defines a set of abelian first class constraints and therefore its Poisson brackets (or commutators in the quantum case) with physical quantities vanish. In the quantum case these constraints are imposed with an appropriate projection on the physical Hilbert space.

The simplest proof of the equivalence between (13) supplemented with (15) and (11) is obtained introducing the δ -function representing the constraint (15) in the path integral with the action (11). By explicit solution of the model we will prove the equivalence in the operatorial formulation.

We can obtain the Poisson brackets by inverting $\tilde{\Omega}$. The result is :

$$\{u_1 u_2\} = u_1 u_2 R_{12}^l, \quad R_{12}^l = \sum_{\alpha \in \Phi} \frac{i}{p_l \alpha} E_\alpha \otimes E_{-\alpha} \quad (16a)$$

$$\{v_1 v_2\} = v_1 v_2 R_{12}^r, \quad R_{12}^r = \sum_{\alpha \in \Phi} \frac{i}{p_r \alpha} E_{-\alpha} \otimes E_\alpha \quad (16b)$$

$$\{\vec{p}_l, u\} = -iu\vec{H}, \quad \{\vec{p}_r, v\} = -i\vec{H}v \quad (16c)$$

The Poisson brackets between left and right variables vanish. The left (right) symmetry generators are given by :

$$J_l^a = itr(t^a u \vec{p}_l \vec{H} u^{-1}) \quad , \quad \{J_l^a, J_l^b\} = f^{abc} J_l^c \quad (17a)$$

$$J_r^a = itr(t^a v^{-1} \vec{p}_r \vec{H} v) \quad , \quad \{J_r^a, J_r^b\} = f^{abc} J_r^c \quad (17b)$$

and the transformations properties of u, v under the group action are :

$$\{J_l^a, u\} = it^a u \quad , \quad \{J_r^a, v\} = ivt^a \quad (18)$$

C. Quantization

Quantization of the previous Poisson brackets (16) gives the following result [10] [11] :

$$u_1 u_2 = u_2 u_1 B_{12}^l, \quad v_1 v_2 = B_{12}^r v_2 v_1 \quad (19a)$$

$$[p_l^i, u] = \hbar u H^i, \quad [p_r^i, v] = \hbar H^i v \quad (19b)$$

$$[p_l^i, p_l^j] = [p_r^i, p_r^j] = 0 \quad (19c)$$

The explicit solution for the B-matrices was obtained in [11] for the case when u , v are matrices in the fundamental representation of $G = SU(n)$. Let Φ denote the space of roots and $\vec{\lambda}_i$ for $i = 1 \dots n$ be the weights in the fundamental representation, with highest weight $\vec{\lambda}_1$. The normalization of these vectors is chosen to be $\vec{\lambda}_i \vec{\lambda}_j = \delta_{ij} - \frac{1}{n}$. The Cartan-Weyl basis is given by :

$$\vec{H} = \sum_i \vec{\lambda}_i e_{ii} \quad (20)$$

and the step operators $E_{\alpha_{ij}}$ are represented by the matrices $(e_{ij})_{ab} = \delta_{ia} \delta_{jb}$ for $\alpha_{ij} = \vec{\lambda}_i - \vec{\lambda}_j \in \Phi$. In this basis the matrices $B_l(B_r)$ have the following form :

$$B_{l,r} = \exp\left(-\sum_{\alpha} \theta_{\alpha}^{l,r} E_{\alpha} \otimes E_{-\alpha}\right) = \sum_{\alpha} (\cos \theta_{\alpha}^{l,r} E_{\alpha} E_{-\alpha} \otimes E_{-\alpha} E_{\alpha} - \sin \theta_{\alpha}^{l,r} E_{\alpha} \otimes E_{-\alpha}) \quad (21a)$$

where $\sin \theta_{\alpha}^l = \frac{\hbar}{p_l \alpha}$ and $\sin \theta_{\alpha}^r = -\frac{\hbar}{p_r \alpha}$

The symmetry generators can be derived from these commutators. Taking into account the corrections due to the ordering problem they have the form :

$$J_l^a = i \text{tr}(t^a(u(p_l H)u^{-1} + 2\hbar H^2)), \quad J_r^a = i \text{tr}(t^a(v^{-1}(p_r H)v + 2\hbar H^2)) \quad (22)$$

The generators (22) fulfil the standard commutation relations :

$$[J_l^a, J_l^b] = f^{abc} J_l^c, \quad [J_r^a, J_r^b] = f^{abc} J_r^c \quad (23)$$

The transformation properties of the u and v operators are defined by :

$$[J_l^a, u] = i\hbar t^a u \quad (24a)$$

$$[J_r^a, v] = i\hbar v t^a \quad (24b)$$

The second-order Casimir operators for left and right theories depend only on p_l^i and p_r^i respectively :

$$C_{l,r} = p_{l,r}^2 - \hbar^2 \text{tr}(H^2 H^2) \quad (25)$$

We can now construct the Hilbert space for these theories. We will consider only the left theory, because the right one is completely analogous :

$$\mathcal{H} = \bigoplus_{\{\lambda\}} \mathcal{H}_\lambda \quad (26)$$

where \mathcal{H}_λ is the irreducible unitary representation with highest weight λ . To describe these spaces it is convenient to introduce the following notations. Every irrep of $SU(n)$ is characterized by a partition :

$$[m] = [m_{in}] = [m_{1n}, m_{2n}, \dots, m_{n-1,n}, 0] \quad (27)$$

of $n - 1$ non-negative integers, obeying the relation :

$$m_{in} \geq m_{i+1,n} \quad (28)$$

An elegant notational convention to label in a unique way a state of the irrep $[m_{in}]$ is the Gelfand pattern [12], denoted by (m) . It consists of a triangular array of $\frac{n(n+1)}{2} - 1$ integers $\{m_{ij}\}$, $1 \leq i \leq j \leq n$ with $m_{nn} = 0$ which satisfy the "betweenness conditions" :

$$m_{i,j+1} \geq m_{i,j} \geq m_{i+1,j} \quad (29)$$

The k -row, ($k \neq n$), of this pattern is a highest weight for some irrep of $U(k)$. The possibility of such a description of states is based on the Weyl branching law for $U(j)$. It asserts that in the restriction from group to subgroup $U(j+1) \rightarrow U(j)$ for any given irrep of $U(j+1)$ with highest weight $[m_{i,j+1}]$ the irrep of $U(j)$ with highest weight $[m_{ij}]$ satisfying the

”betweenness condition” (29) occurs only once . Therefore the Gelfand pattern corresponds to the branching chain of restrictions from group to subgroup :

$$SU(n) \rightarrow U(n-1) \rightarrow \cdots \rightarrow U(1) \quad (30)$$

We define now the action of the operators in the chiral theory on the Hilbert space. The operators p^i commute with all the generators J^a . Therefore they depend only on the highest weight. Their explicit action on the state is :

$$p^i |(m)\rangle = \hbar(m_{in} + n - i) |(m)\rangle \quad (31)$$

We construct now the action of the u_{ij} operators . From equation (24) we see that the u_{ij} ’s are tensor operators in the fundamental representation and the i index designates the magnetic quantum number.

From the commutation relations :

$$[p^i, u] = \hbar u H^i \quad (32)$$

it follows that

$$\langle (\tilde{m}) | u_{ij} | (m) \rangle \sim \prod_{k=1}^{n-1} \delta_{\tilde{m}_{kn}, m_{kn} + \delta_{kj} - \delta_{kn}} \quad (33)$$

Therefore the meaning of the right index of u_{ij} is completely different from the meaning of the left one : u_{ij} carries an arbitrary vector belonging to irrep $[m]_n$ into irrep $[\tilde{m}]_n$:

$$\tilde{m}_{kn} = m_{kn} + \delta_{kj} - \delta_{jn} \quad (34)$$

We will relate the u_{ij} to the unit tensor operators of $SU(n)$ [13]. The unit tensor operator is labeled by two Gelfand patterns, denoted by :

$$\left\langle \begin{array}{c} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{array} \right\rangle \quad (35)$$

In this notation the lower pattern

$$(M)_n \equiv \begin{pmatrix} [M]_n \\ (M)_{n-1} \end{pmatrix} \quad (36)$$

identifies the transformation properties in $SU(n)$ of the tensor operator. The upper pattern :

$$(\Gamma)_n \equiv \begin{pmatrix} (\Gamma)_{n-1} \\ (M)_n \end{pmatrix} \quad (37)$$

is of the same form as a Gelfand pattern and satisfies the same "betweenness conditions" :

$$\Gamma_{i,j+1} \geq \Gamma_{ij} \geq \Gamma_{i+1,j+1} \quad (38)$$

$(\Gamma)_n$ designates the fact that the tensor operator carries an arbitrary vector, belonging to irrep $[m]_n$ into a vector, belonging to irrep $[\tilde{m}]_n$ of $SU(n)$, where :

$$\tilde{m}_{in} = m_{in} + \Delta_{in}(\Gamma) \quad (39)$$

$$\Delta_{in}(\Gamma) = \sum_{j=1}^i \Gamma_{ji} - \sum_{j=1}^{i-1} \Gamma_{j,i-1} - \sum_{j=1}^n m_{jn} + \sum_{j=1}^{n-1} m_{j,n-1}$$

$$[\Delta(\Gamma)]_n \equiv [\Delta_{in}(\Gamma) \cdots 0]$$

The restriction chain (30) has an important consequence for the unit tensor operators (35).

It allows to define its action on a given state in a recursive way :

$$\left\langle \begin{pmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{pmatrix} \right| \begin{pmatrix} [m]_n \\ [m]_{n-1} \\ (m)_{n-2} \end{pmatrix} \right\rangle = \sum_{(\gamma)_{n-2}} \left[\begin{pmatrix} (\Gamma)_{n-1} & [m]_n \\ [M]_n & \\ (\gamma)_{n-1} & [m]_{n-1} \end{pmatrix} \right] \left\langle \begin{pmatrix} (\gamma)_{n-2} \\ [M]_{n-1} \\ (M)_{n-2} \end{pmatrix} \right| \begin{pmatrix} [m]_n + [\Delta(\Gamma)]_n \\ [m]_{n-1} \\ (m)_{n-2} \end{pmatrix} \right\rangle \quad (40)$$

where

$$(\gamma)_{n-1} \equiv \begin{pmatrix} [M]_{n-1} \\ (\gamma)_{n-2} \end{pmatrix}$$

Here the $\left\langle \begin{pmatrix} (\gamma)_{n-2} \\ [M]_{n-1} \\ (M)_{n-2} \end{pmatrix} \right\rangle$ operator in the r. h. s. is the $U(n-1)$ unit tensor operator, acting

on $|(m)_{n-1}\rangle$ which is the state from the irrep of $U(n-1)$ with highest weight $[m]_{n-1}$. The remaining part in (40) :

$$\begin{bmatrix} (\Gamma)_{n-1} & [m]_n \\ [M]_n \\ (\gamma)_{n-1} & [m]_{n-1} \end{bmatrix} \quad (41)$$

are c-numbers which depend on the $(\Gamma)_{n-1}, [M]_n, (\gamma)_{n-1}$ and on the first two rows of $|(m)\rangle$: $[m]_n$ and $[m]_{n-1}$. Such a decomposition has many interesting properties [13] and it will be useful in the following .

The non-zero matrix elements of the unit tensor operators can be understood as the Clebsch-Gordan coefficients of $SU(n)$. We can couple two state vectors $|(m)\rangle$ and $|(M)\rangle$ from the space of direct product of irreps $[m]_n$ and $[M]_n$ to obtain coupled state vectors which are again the Gelfand basis vectors for an irrep of $SU(n)$:

$$\begin{aligned} & \left| \begin{pmatrix} [m] + [\Delta(\Gamma)] \\ (m') \end{pmatrix}; (\Gamma) \right\rangle = \\ & \sum_{(M)(m)} \left\langle \begin{pmatrix} [m] + [\Delta(\Gamma)] \\ (m') \end{pmatrix} \right| \left\langle \begin{pmatrix} \Gamma \\ [M] \\ (M) \end{pmatrix} \right| \left| \begin{pmatrix} m \\ (m) \end{pmatrix} \right\rangle \left| \begin{pmatrix} M \\ (M) \end{pmatrix} \right\rangle \times \left| \begin{pmatrix} m \\ (m) \end{pmatrix} \right\rangle \end{aligned} \quad (42)$$

Because the operators u_{ij} are tensor operators in the fundamental representation, we will need the explicit form of (35) only for $[M]_n = [10 \cdots 0]$. For these tensor operators and for the corresponding numbers (41) in the decomposition (40) it is possible to introduce the simplified notations :

$$\begin{aligned} & \left\langle \begin{pmatrix} (\Gamma)_{n-1} \\ [10 \cdots 0] \\ (M)_{n-1} \end{pmatrix} \right\rangle \rightleftharpoons \left\langle \begin{pmatrix} i \\ j \end{pmatrix} \right\rangle \end{aligned} \quad (43a)$$

$$\begin{aligned} & \begin{bmatrix} (\Gamma)_{n-1} & [m]_n \\ [M]_n \\ (M)_{n-1} & [m]_{n-1} \end{bmatrix} \rightleftharpoons \begin{bmatrix} i & [m]_n \\ j & [m]_{n-1} \end{bmatrix} \end{aligned} \quad (43b)$$

The r. h. s. of (43a) denotes the (unique) tensor operator with upper and lower pattern defined by the shift (39) $\Delta_n(i)$, $(\Delta_n(j))$ respectively . For $i \neq n$ we have :

$$\Delta_n(i) = [0 \cdots 010 \cdots 0] \quad , \quad \Delta_n(j) = [0 \cdots 010 \cdots 0] \quad (44)$$

where 1 appears in the position $i(j)$. For $i = n$ we have :

$$\Delta_n(n) = [-1 - 1 \cdots - 10] \quad , \quad \Delta_n(n) = [-1 - 1 \cdots - 10 - 1 \cdots - 1] \quad (45)$$

where 0 appears in the position j . The numbers $\begin{bmatrix} i & [m]_n \\ j & [m]_{n-1} \end{bmatrix}$ can be found explicitly [13] :

$$\begin{bmatrix} i & [m]_n \\ j & [m]_{n-1} \end{bmatrix}^2 = \frac{\prod_{j' \neq j}^{n-1} (p_{in} - p_{j',n-1}) \prod_{i' \neq i}^n (p_{j,n-1} - p_{i',n} + 1)}{\prod_{i' \neq i}^n (p_{in} - p_{i',n}) \prod_{j' \neq j}^{n-1} (p_{j,n-1} - p_{j',n-1} + 1)} \quad (46)$$

where $p_{in} = m_{in} + n - i$; $p_{nn} \equiv 0$. The coefficients in (46) are chosen in such a way that u_{ij} operators correspond to the matrix elements of the unitary matrix :

$$\sum_j u_{jk}^+ u_{ij} = \delta_{ik} \quad (47a)$$

$$\sum_j u_{jk} u_{ij}^+ = \hat{I}_j \delta_{ik} \quad (47b)$$

where $\hat{I}_j |(m)\rangle = 0$ if $u_{ij}^+ |(m)\rangle = 0$, otherwise $\hat{I}_j |(m)\rangle = |(m)\rangle$.

In the Appendix 1 it is shown that the commutation relations of the unit tensor operators in the fundamental representation coincide with the B matrix relations (19), (21) for the u_{ij} operators. Therefore the explicit realization of the u_{ij} operators is

$$u_{ji} = \left\langle \begin{matrix} i \\ j \end{matrix} \right\rangle \quad (48)$$

D. Combining the left and right theories.

After quantizing the chiral part separately we return now to our original model (1). As discussed in section 2 B the physical Hilbert space \mathcal{H}_{phys} will be the direct product of chiral Hilbert spaces on which we have imposed the "Gauss law" :

$$\forall |\psi\rangle \in \mathcal{H}_{phys} \quad (\vec{p}_l - \vec{p}_r)|\psi\rangle = 0 \quad (49)$$

It means that :

$$\mathcal{H}_{phys} = \bigoplus_{\{\lambda\}} \mathcal{H}_{\lambda}^l \otimes \mathcal{H}_{\lambda}^r \quad (50)$$

where $\{\lambda\}$ is the set of all irreps of $SU(n)$ with highest weights λ and the left and right irreps have the same highest weights. Therefore the states in the physical Hilbert space are labeled by λ and the magnetic quantum numbers $(\gamma)_{n-1}, (\chi)_{n-1}$. In the Gelfand pattern notation the states in \mathcal{H}_{phys} can be denoted by a double Gelfand pattern :

$$\left| \begin{array}{c} (\gamma)_{n-1} \\ [m]_n \\ (\chi)_{n-1} \end{array} \right\rangle \quad (51)$$

where the upper (lower) parts of this double pattern designate the left (right) state respectively. It is isomorphic to the Hilbert space in the standard treatment discussed in the section 2 A, if the matrix elements of the representation are labeled by the highest weight $[m]_n$ and the magnetic quantum numbers $(\gamma)_{n-1}, (\chi)_{n-1}$.

We discuss now the physical operators in the theory. All physical operators must commute with $\vec{\phi}$. The operator corresponding to the g -operator in the initial notations is given by :

$$g_{ij} = \sum_{k=1}^n u_{ik} v_{kj} \quad (52)$$

It is easy to check that :

$$[g_{ij}, (\vec{p}_l - \vec{p}_r)] = 0 \quad (53)$$

Therefore we can consistently restrict g to the physical Hilbert space. We check now that in the restricted Hilbert space :

$$[g_1, g_2] = 0 \quad (54)$$

Indeed it follows from (19), (21) that :

$$g_{ij}g_{kl} = \sum_{qs} u_{iq}v_{qj}u_{ks}v_{sl} = \quad (55)$$

$$\sum_{qs} (u_{kq}u_{is}B_{qs}^{l1} + u_{ks}u_{iq}B_{qs}^{l2})(B_{qs}^{r1}v_{ql}v_{sj} + B_{qs}^{r2}v_{sl}v_{qj}) =$$

$$\sum_{qs} (u_{kq}u_{is}v_{ql}v_{sj}(B_{qs}^{l1}B_{qs}^{r1} + B_{qs}^{l2}B_{qs}^{r2}) + u_{kq}u_{is}v_{sl}v_{qj}(B_{qs}^{l1}B_{qs}^{r2} + B_{sq}^{l2}B_{sq}^{r1}))$$

where e. g.

$$B_{qs}^{l1} = \frac{\hbar}{p_{lq} - p_{ls}}, \quad B_{qs}^{l2} = \sqrt{1 - \left(\frac{\hbar}{p_{lq} - p_{ls}}\right)^2}$$

and similar expressions for $B_{qs}^{r1,2}$. The definition of the physical Hilbert space and the antisymmetry of $B_{qs}^{(l,r)1}$ with respect to the permutation $(qs) \rightarrow (sq)$ prove the statement.

We can calculate explicitly the action of the g operator on the ground state using (52) :

$$g_{ij} \left| \begin{array}{c} (0)_{n-1} \\ [0]_n \\ (0)_{n-1} \end{array} \right\rangle = \left| \begin{array}{c} (\gamma)_{n-1} \\ [10 \cdots 0]_n \\ (\chi)_{n-1} \end{array} \right\rangle \quad (56)$$

where

$$\Delta_n(\gamma) = [0 \cdots 01^i 0 \cdots 0]$$

$$\Delta_n(\chi) = [0 \cdots 01_j 0 \cdots 0]$$

Equation (56) reproduces the result in the standard treatment [9]. The Hamiltonian of the full system is given by :

$$\mathcal{H} = \frac{i}{2} \sum_{k=1}^r p_k p_k - \hbar^2 \text{tr}(H^2 H^2) \quad , \quad \text{where} \quad p_k = \frac{1}{2}(p_k^l + p_k^r) \quad (57)$$

and it coincides with the second-order Casimir operator. Using (56) and (57) we recover the propagator (3).

3. CHIRAL QUANTIZATION OF THE WZW MODEL

A. Classical analysis in Chiral Variables

We return now to our main problem , the chiral quantization of the WZW model. Let us consider the group-valued field $g(\tau, x)$, where x is the spatial coordinate for a circle with unit radius , and $g(\tau, x + 2\pi) = g(\tau, x)$. Let ψ denote the longest root. The action for the WZW model can be written as [4] :

$$S[g] = -\frac{k\psi^2}{16\pi} \int_{\mathcal{M}} tr(\partial_\mu g \partial^\mu g^{-1}) d^2x + \Gamma[g] \quad (58)$$

where $\Gamma[g]$ is the topological term :

$$\Gamma[g] = \frac{k\psi^2}{24\pi} \int_{\mathcal{B}} d^3X \epsilon^{\alpha\beta\gamma} tr(\tilde{g}^{-1} \partial_\alpha \tilde{g} \tilde{g}^{-1} \partial_\beta \tilde{g} \tilde{g}^{-1} \partial_\gamma \tilde{g}) \quad (59)$$

Here \mathcal{B} is the three dimensional domain whose boundary is the cylinder \mathcal{M} and $\tilde{g}(\tau, x, y) \in G$ is the map from \mathcal{B} to G such that $\tilde{g}(\tau, x, 1) = g(\tau, x)$. The singlevaluedness of the probability amplitude requires k to be an integer.

The equations of motion following from the action (58) are :

$$\partial_+(g^{-1} \partial_- g) = \partial_-(\partial_+ g g^{-1}) = 0 \quad , \quad x^\pm \equiv \tau \pm x$$

The general solution of these equations is :

$$g(\tau, x) = \tilde{u}(x^+) (\exp 2i\vec{p}\vec{H}(x^+ + x^-)) \tilde{v}(x^-)$$

This solution is invariant under the transformation :

$$\tilde{u} \rightarrow \tilde{u}h, \quad \tilde{v} \rightarrow h^{-1}\tilde{v} \quad , \quad h \in T \quad (60)$$

like the solution of the finite-dimensional model.

Using the fact that any solution is completely defined by the given $g(\tau, x)|_{\tau=0}$ and $\partial_\tau g(\tau, x)|_{\tau=0}$ let us change variables in analogy with what we did in section 2 [14] :

$$g(\tau, x)|_{\tau=0} = \tilde{u}\tilde{v} \quad (61)$$

$$g^{-1}\partial_\tau g|_{\tau=0} = \tilde{v}^{-1}\tilde{u}^{-1}\partial_x\tilde{u}\tilde{v} + 4i\tilde{v}^{-1}\vec{p}\vec{H}\tilde{v} - \tilde{v}^{-1}\partial_x\tilde{v}$$

where the periodic variables \tilde{u}, \tilde{v} belong to the loop group LG and we used the following identity :

$$\partial_\tau\tilde{u}|_{\tau=0} = \partial_x\tilde{u} \quad ; \quad \partial_\tau\tilde{v}|_{\tau=0} = -\partial_x\tilde{v} \quad (62)$$

In these variables the action takes the form :

$$\begin{aligned} S = \frac{k\psi^2}{8\pi} \{ \int_{\mathcal{M}} tr(\tilde{u}^{-1}\partial_x\tilde{u}\tilde{u}^{-1}\partial_t\tilde{u} - \tilde{v}^{-1}\partial_x\tilde{v}\tilde{v}^{-1}\partial_t\tilde{v} + \\ + 4i\vec{p}\vec{H}(\tilde{u}^{-1}\partial_t\tilde{u} + \partial_t\tilde{v}\tilde{v}^{-1} - \tilde{u}^{-1}\partial_x\tilde{u} + \partial_x\tilde{v}\tilde{v}^{-1}) \\ + 8(\vec{p}\vec{H})^2 - (\tilde{u}^{-1}\partial_x\tilde{u})^2 - (\tilde{v}^{-1}\partial_x\tilde{v})^2)dxdt \} + \Gamma[\tilde{u}] + \Gamma[\tilde{v}] \end{aligned} \quad (63)$$

where $\Gamma[\tilde{u}]$ and $\Gamma[\tilde{v}]$ are the topological terms.

This action and the relevant symplectic form are degenerate due to the invariance (60). The situation is completely analogous to the "toy" model (6) . In a similar way we introduce separate \vec{p}_l and \vec{p}_r and the constraints (15). The new action is :

$$\begin{aligned} S = \frac{k\psi^2}{8\pi} \{ \int_{\mathcal{M}} tr(\tilde{u}^{-1}\partial_x\tilde{u}\tilde{u}^{-1}\partial_t\tilde{u} - \tilde{v}^{-1}\partial_x\tilde{v}\tilde{v}^{-1}\partial_t\tilde{v} + 4i\vec{p}_l\vec{H}(\tilde{u}^{-1}\partial_t\tilde{u} - \tilde{u}^{-1}\partial_x\tilde{u}) + \\ 4i\vec{p}_r\vec{H}(\partial_t\tilde{v}\tilde{v}^{-1} + \partial_x\tilde{v}\tilde{v}^{-1}) + 2((\vec{p}_l + \vec{p}_r)\vec{H})^2 - (\tilde{u}^{-1}\partial_x\tilde{u})^2 - (\tilde{v}^{-1}\partial_x\tilde{v})^2)dxdt \} + \\ + \Gamma[\tilde{u}] + \Gamma[\tilde{v}] \end{aligned} \quad (64)$$

The corresponding symplectic form is :

$$\begin{aligned} \Omega = \frac{k\psi^2}{8\pi} \int tr(\tilde{u}^{-1}\delta\tilde{u}\partial_x(\tilde{u}^{-1}\delta\tilde{u}) + 4i\vec{p}_l\vec{H}\tilde{u}^{-1}\delta\tilde{u}\tilde{u}^{-1}\delta\tilde{u} - d\vec{p}_l\vec{H}\tilde{u}^{-1}\delta\tilde{u} - \\ - \delta\tilde{v}\tilde{v}^{-1}\partial_x(\delta\tilde{v}\tilde{v}^{-1}) + 4i\vec{p}_r\vec{H}\delta\tilde{v}\tilde{v}^{-1}\delta\tilde{v}\tilde{v}^{-1} - d\vec{p}_r\vec{H}\delta\tilde{v}\tilde{v}^{-1}) \end{aligned} \quad (65)$$

It is convenient to introduce the new variables :

$$u(x) = \tilde{u}(x) \exp 2i\vec{p}_l \vec{H} x \quad , \quad v(x) = \tilde{v}(x) \exp -2i\vec{p}_r \vec{H} x \quad (66)$$

The symplectic form (65) is non-degenerate and we can obtain the Poisson brackets :

$$\{u(x), \vec{p}_l\} = \frac{\beta}{2} u(x) \vec{H} \quad , \quad \{v(x), \vec{p}_r\} = \frac{\beta}{2} \vec{H} v(x) \quad (67)$$

$$\{u_1(x), u_2(y)\} = u_1(x) u_2(y) R_l(x - y)$$

$$\{v_1(x), v_2(y)\} = R_r(x - y) v_1(x) v_2(y)$$

where

$$R_l(x) = \frac{\beta}{2} \eta(x) \sum_{j=1}^r H^j \otimes H^j + i \frac{\beta}{2} \sum_{\alpha} \frac{1}{\sin(\alpha \cdot p_l)} e^{-i\alpha \cdot p_l \eta(x)} E_{\alpha} \otimes E_{-\alpha} \quad (68a)$$

$$R_r(x) = \frac{\beta}{2} \eta(x) \sum_{j=1}^r H^j \otimes H^j - i \frac{\beta}{2} \sum_{\alpha} \frac{1}{\sin(\alpha \cdot p_r)} e^{-i\alpha \cdot p_r \eta(x)} E_{\alpha} \otimes E_{-\alpha} \quad (68b)$$

$$\beta = \frac{4\pi}{\psi^2 k}$$

and $\eta(x) = 2[\frac{x}{2\pi}] + 1$, $[x]$ denotes the maximal integer , less then x . The Poisson brackets between left and right variables vanish. Since the Poisson brackets (67) are derived from a non-degenerate lagrangian they fulfil automatically the Jacobi identities.

From these Poisson brackets it is possible to derive the classical transformation properties of $u(x)$ and $v(x)$ under Kac-Moody symmetry action :

$$\{J_l^a(x), u(y)\} = it^a u(y) \delta(x - y) \quad (69a)$$

$$\{J_r^a(x), v(y)\} = iv(y) t^a \delta(x - y) \quad (69b)$$

where the Kac-Moody currents are given by :

$$J_l(x) = \frac{ik}{4\pi} \partial_x u u^{-1} \quad , \quad J_r(x) = \frac{ik}{4\pi} v^{-1} \partial_x v \quad (70)$$

B. The quantization of the chiral theory

We quantize these Poisson brackets , according [5] [6] [7] [8]. In the following we will map the cylinder to the complex plane $e^{ix} \rightarrow z$. From now on we will consider only the $\hat{SU}(n)_k$ WZW model , with the u and v operators in the fundamental representation. The algebra of the quantum operators is :

$$[\vec{p}_l, u(z)] = -\frac{\beta}{2}\hbar u(z)\vec{H} \quad , \quad [\vec{p}_l, v(\bar{z})] = -\frac{\beta}{2}\hbar\vec{H}v(\bar{z}) \quad (71a)$$

$$u_1(z_1)u_2(z_2) = u_2(z_2)u_1(z_1)B_l\left(\frac{z_1}{z_2}\right) \quad , \quad v_1(\bar{z}_1)v_2(\bar{z}_2) = B_r\left(\frac{\bar{z}_1}{\bar{z}_2}\right)v_2(\bar{z}_2)v_1(\bar{z}_1) \quad (71b)$$

$$J_l^a(z_1)u(z_2) = \frac{it^a u(z_2)}{z_1 - z_2} + \dots \quad , \quad J_r^a(\bar{z}_1)v(\bar{z}_2) = \frac{iv(\bar{z}_2)t^a}{\bar{z}_1 - \bar{z}_2} + \dots \quad (71c)$$

where $B_{l,r}(z)$ are the braiding matrices , which can be obtained by exponentiating the classical R -matrix , like (21). It is enough to consider only the left-invariant theory , because the right one is completely analogous.

The explicit expression for $B_l(z)$ is :

$$\begin{aligned} B_l(z) = & q^{(\frac{n-1}{n})\eta(\arg(z))} \left(1 \otimes 1 - \sum_{\alpha \in \Phi} E_\alpha E_{-\alpha} \otimes E_{-\alpha} E_\alpha \right) \\ & + q^{-\frac{1}{n}\eta(\arg(z))} \sum_{\alpha \in \Phi} \cos \theta(\alpha \cdot p_l) E_\alpha E_{-\alpha} \otimes E_{-\alpha} E_\alpha \\ & - q^{-\frac{1}{n}\eta(\arg(z))} \sum_{\alpha \in \Phi} e^{-i\alpha \cdot p_l} \sin \theta(\alpha \cdot p_l) E_\alpha \otimes E_{-\alpha} \end{aligned} \quad (72)$$

$q \equiv e^{\frac{i\pi}{k+n}}$

where

$$\sin \theta(\alpha \cdot p_l) \equiv \frac{\sin \frac{\pi}{k+n}}{\sin \alpha \cdot p_l}$$

The expression for β including the quantum correction is :

$$\beta\hbar = \frac{2i\pi}{k+n} \quad (73)$$

The braiding matrix can be related to the Racah matrix of $U_{2q}(SL(n))$ in the fundamental representation.

We assume that the Hilbert space of the system is :

$$\mathcal{H} = \bigoplus_{\{\lambda\}} \mathcal{H}_\lambda \quad (74)$$

where $\{\lambda\}$ denotes the set of all integrable representations of $\hat{SU}(n)_k$. The integrability condition for the $\hat{SU}(n)_k$ [15] is :

$$k \geq m_{1n} \quad (75)$$

We will find an explicit representation of all chiral operators on this Hilbert space. Because the p_i^j operators commute with all Kac-Moody currents, they depend only on the highest weights of the representations and we can define :

$$p_i^j |[m]_n, (m)\rangle = \frac{\hbar}{k+n} (m_{in} + n - i) |[m]_n, (m)\rangle \quad (76)$$

where $[m]_n$ denotes the highest weight of some integrable representation and (m) denotes all other quantum numbers characterizing the state. The commutation relations of $u_{ij}(z)$ and $J_n^{(\ell)a}$ imply that $u_{ij}(z)$ is a primary field in the fundamental representation. For its matrix elements between the zero level states one has :

$$\Psi_1 = \langle (\tilde{m})_n^\infty | u_{ij}(z) | (m)_n^0 \rangle = \frac{Inv(V_{\tilde{m}}^* \otimes V \otimes V_m)}{z^{-h(\tilde{m})+h+h(m)}} \quad (77)$$

where $Inv(V_{\tilde{m}}^* \otimes V \otimes V_m)$ is a $SU(n)$ invariant functional on $V_{\tilde{m}}^* \otimes V \otimes V_m$; $V_{\tilde{m}}, V_m$ are the irreps of $SU(n)$ with the highest weights $[\tilde{m}]_n, [m]_n$ respectively and V is the fundamental representation. The $h(m)$ is the conformal dimension of the operator :

$$h(m) = \frac{C_2(m)}{k+n} \quad (78)$$

and $C_2(m)$ is the value of the second-order Casimir operator. All other matrix elements can be computed using the chiral symmetry. Equations (71a), (76) imply that $\Psi_1(z)$ is not equal to zero only if the highest weights are related by :

$$[\tilde{m}]_n = [m]_n + \Delta_n(j) \quad (79)$$

Equation (79) and the interpretation of the matrix elements of the unit tensor operators (42) means that we can write $\Psi_1(z)$ as follows :

$$\Psi_1(z) = \frac{\langle (\tilde{m})_n | \gamma_{ij} | (m)_n \rangle}{z^{-h(\tilde{m})+h+h(m)}} C_{[m]_n, j} \quad (80)$$

where γ_{ij} is the unit tensor operator , defined earlier (43a), (46) , and $C_{[m]_n, j}$ is a constant depending only on j and the highest weight $[m]_n$. The conditions (79), (80) mean that the right index of the chiral vertex operator $u_{ij}(z)$ determines the difference between integrable highest weights $[m]_n$ and $[\tilde{m}]_n$ like the right index of the unit tensor operators γ_{ij} . Using the projected vertex operators $\Phi_{\tilde{m}m}^k(z)$ [2] [3] one can write the $u_{ij}(z)$ as :

$$u_{ij}(z) = \sum_{[\tilde{m}], [m]} \Phi_{\tilde{m}m}^i(z) C_{[m]_n, j} \delta_{[\tilde{m}]_n, [m]_n + \Delta_n(j)} \quad (81)$$

where the $\Phi_{\tilde{m}m}^i$ is in the fundamental representation with magnetic quantum number i and the sum is over all integrable highest weights $[\tilde{m}]_n, [m]_n$.

The commutation relations (71b) imply restrictions on the constants $C_{[m]_n, j}$. To determine these restrictions we need to find the matrix elements of the product of the two u operators between arbitrary states. However it is enough to consider only the matrix elements of this product between the zero level states, since the B matrix (72) commutes with the Kac-Moody currents :

$$\Psi_2(z_1, z_2) = \langle (\tilde{m})_n^\infty | u_{ij}(z_1) u_{kl}(z_2) | (m)_n^0 \rangle \quad (82)$$

In the discussion of the tensorial properties of (82) we can use the results for the "toy model" because the level zero subspace in the representation of $\hat{S}U(n)_k$ is equivalent to the irreducible representation of $SU(n)$. To determine the z_1, z_2 dependence of (82) it is convenient to use the KZ equations [16] in the form proposed in [17].

The tensorial structure of (82) is given by the condition :

$$\Psi_2(z_1, z_2) \in Inv(V_m^* \otimes V \otimes V \otimes V_m) \quad (83)$$

The dimension of $Inv(V_{\tilde{m}}^* \otimes V \otimes V \otimes V_m)$ is different from zero only if :

$$[\tilde{m}]_n = [m]_n + \Delta_n(j) + \Delta_n(l) \quad (84)$$

We will treat separately the cases : $j \neq l$ and $j = l$.

Let us first consider the $j \neq l$ case. The space of invariant couplings for the given $|(\tilde{m})\rangle \in V_{\tilde{m}}$, $|(m)\rangle \in V_m$ is two dimensional. It is convenient to choose as a basis in this space the following vectors :

$$T_1 = \langle(\tilde{m})|\gamma_{ij}\gamma_{kl}|(m)\rangle \quad ; \quad T_2 = \langle(\tilde{m})|\gamma_{kj}\gamma_{il}|(m)\rangle \quad (85)$$

where γ_{pq} is the unit tensor operator.

In these notations $\Psi_2(z_1, z_2)$ takes the form :

$$\Psi_2(z_1, z_2) = T_1\psi_1(z_1, z_2) + T_2\psi_2(z_1, z_2) \quad (86)$$

where $\psi_{1,2}$ depends only on j, l and the highest weights $[\tilde{m}]_n, [m]_n$. Following [17] we can write the KZ equations for $\Psi_2(z_1, z_2)$ in the form :

$$(k+n)\frac{\partial\Psi_2}{\partial z_1} = \left(\frac{t_1 \otimes t_2}{z_1 - z_2} + \frac{t_1 \otimes t_m}{z_1}\right)\Psi_2 \quad (87a)$$

$$(k+n)\frac{\partial\Psi_2}{\partial z_2} = \left(\frac{t_2 \otimes t_1}{z_2 - z_1} + \frac{t_2 \otimes t_m}{z_2}\right)\Psi_2 \quad (87b)$$

where t_i are the generators of $SU(n)$ in the i representation.

Using the properties of the unit tensor operators γ_{pq} we get (see Appendix 2) :

$$(t_1 \otimes t_2)T_1 = T_2 - \frac{1}{n}T_1 \quad , \quad (t_1 \otimes t_2)T_2 = T_1 - \frac{1}{n}T_2 \quad (88a)$$

$$(t_1 \otimes t_m)T_1 = -T_2 + \frac{1}{n}T_1 + \frac{1}{2}(C_2([\tilde{m}]_n) - C_2([m]_n + \Delta_n(l)) - C_2)T_1 \quad (88b)$$

$$(t_2 \otimes t_m)T_2 = -T_1 + \frac{1}{n}T_2 + \frac{1}{2}(C_2([\tilde{m}]_n) - C_2([m]_n + \Delta_n(l)) - C_2)T_2 \quad (88c)$$

$$(t_2 \otimes t_m)T_1 = \frac{1}{2}(C_2([m]_n + \Delta_n(l)) - C_2([m]_n) - C_2)T_1 \quad (88d)$$

$$(t_1 \otimes t_m)T_2 = \frac{1}{2}(C_2([m]_n + \Delta_n(l)) - C_2([m]_n) - C_2)T_2 \quad (88e)$$

After the substitution :

$$\psi_{1,2}(z_1, z_2) = (z_1 z_2)^{\frac{C_2([\tilde{m}]_n) + C_2([m]_n) - 2C_2}{4(k+n)}} \chi_{1,2}(z) \quad (89)$$

where $z = \frac{z_2}{z_1}$ we get the system of ordinary differential equations :

$$\frac{d}{dz} \begin{pmatrix} \chi_1(z) \\ \chi_2(z) \end{pmatrix} = \left\{ \frac{1}{z} \begin{pmatrix} -\frac{1}{2}(\mu + \beta) & -\gamma \\ 0 & \frac{1}{2}(\mu + 3\beta) \end{pmatrix} + \frac{1}{1-z} \begin{pmatrix} \beta & -\gamma \\ -\gamma & \beta \end{pmatrix} \right\} \begin{pmatrix} \chi_1(z) \\ \chi_2(z) \end{pmatrix} \quad (90)$$

where

$$\mu = \frac{C_2([\tilde{m}]_n) + C_2([m]_n) - 2C_2([m]_n + \Delta_n(l))}{2(k+n)} - \frac{1}{n(k+n)}$$

$$\beta = \frac{1}{n(k+n)} \quad ; \quad \gamma = \frac{1}{k+n}$$

Using the explicit expression for $C_2([m]_n)$:

$$C_2([m]_n) = \sum_{i=1}^{n-1} (m_{in})^2 - \frac{1}{n} \left(\sum_{i=1}^{n-1} m_{in} \right)^2 + \sum_{i < j} (m_{in} - m_{jn})$$

we obtain :

$$\mu = \frac{m_{jn} - m_{ln} + l - j}{k+n} \quad (91)$$

The equations (90) are of Gaussian type. Their general solution is :

$$\chi_1^+ = C^+(1-z)^{-\beta+\gamma} z^{-\frac{1}{2}(\mu-\beta)} F(\gamma, 1+\gamma-\mu, 1-\mu, z) \quad (92a)$$

$$\chi_1^- = C^-(1-z)^{-\beta+\gamma} z^{\frac{1}{2}(\mu+\beta)} F(1+\gamma, \gamma+\mu, 1+\mu, z) \quad (92b)$$

$$\chi_2^+ = C^+ \frac{\gamma}{\mu-1} (1-z)^{-\beta+\gamma} z^{1-\frac{1}{2}(\mu-\beta)} F(1+\gamma, 1+\gamma-\mu, 2-\mu, z) \quad (92c)$$

$$\chi_2^- = -C^- \frac{\mu}{\gamma} (1-z)^{-\beta+\gamma} z^{\frac{1}{2}(\mu+\beta)} F(\gamma, \gamma+\mu, \mu, z) \quad (92d)$$

where $F(\alpha, \beta, \gamma, z)$ are the hypergeometric functions. The functional form of these solutions coincides with the one obtained from the fusion rules argumentation [18]. Let us notice also that if we take

$$[m]_n = [11 \cdots 1 \cdots 10]$$

$$[\tilde{m}]_n = [10 \cdots 0 \cdots 00] \quad ; \quad j = 1 \quad , \quad l = n$$

we obtain the KZ solution [16].

From the expression for the matrix elements of $u_{ij}(z)$ between the zero level states we expect that for $z \rightarrow 0$:

$$\begin{aligned} \lim_{z \rightarrow 0} \Psi_2(z) &= T_1 C_{[m]_n + \Delta_n(l), j} C_{[m]_n, l} z^{C_2([m]_n + \Delta_n(l)) - C_2([m]_n) - C_2} \\ &+ O(z^{C_2([m]_n + \Delta_n(l)) - C_2([m]_n) - C_2 + 1}) \end{aligned} \quad (93)$$

It means that :

$$\Psi_2(z_1, z_2) = (z_1 z_2)^{\frac{C_2([\tilde{m}]_n) + C_2([m]_n) - 2C_2}{4(k+n)}} (T_1 \chi_1^+(z) + T_2 \chi_2^+(z)) \quad (94)$$

For C^+ we get :

$$C^+ = C_{[m]_n + \Delta_n(l), j} \times C_{[m]_n, l} \quad (95)$$

We can also write down the expressions for :

$$\Phi_2(z_1, z_2) = \langle (\tilde{m})^\infty | u_{kj}(z_1) u_{il}(z_2) | (m)^0 \rangle = (z_1 z_2)^\Delta (T_2 \chi_1^+(z) + T_1 \chi_2^+(z)) \quad (96a)$$

$$\Xi_2(z_1, z_2) = \langle (\tilde{m})^\infty | u_{kl}(z_1) u_{ij}(z_2) | (m)^0 \rangle = (z_1 z_2)^\Delta (T_3 \xi_1^+(z) + T_4 \xi_2^+(z)) \quad (96b)$$

where $\Delta = \frac{C_2([\tilde{m}]_n) + C_2([m]_n) - 2C_2}{4(k+n)}$.

The tensorial part of the solution $\Xi(z_1, z_2)$ is given by :

$$T_3 = \langle (\tilde{m}) | \gamma_{kl} \gamma_{ij} | (m) \rangle \quad (97a)$$

$$T_4 = \langle (\tilde{m}) | \gamma_{kj} \gamma_{il} | (m) \rangle \quad (97b)$$

The z -dependence of $\Xi(z_1, z_2)$ can be obtained simply by changing the sign of μ in (94) :

$$\xi_1(z) = \tilde{C} (1-z)^{-\beta+\gamma} z^{\frac{1}{2}(\beta+\mu)} F(\gamma, 1+\gamma+\mu, 1+\mu, z) \quad (98a)$$

$$\xi_2(z) = -\tilde{C} \frac{\gamma}{1+\mu} (1-z)^{-\beta+\gamma} z^{1+\frac{1}{2}(\beta+\mu)} F(1+\gamma, 1+\gamma+\mu, 2+\mu, z) \quad (98b)$$

The behavior of the $\xi_{1,2}(z)$ near the point $z = 0$ implies that :

$$\tilde{C} = C_{[m]_n + \Delta_n(j), l} \times C_{[m]_n, j} \quad (99)$$

It can be shown (see Appendix 3) that :

$$\xi_1(z) = -\frac{\tilde{C}}{C^-} \left(\frac{\gamma^2}{\mu^2 - \gamma^2} \chi_1^-(z) + \frac{\gamma\mu}{\mu^2 - \gamma^2} \chi_2^-(z) \right) \quad (100a)$$

$$\xi_2(z) = -\frac{\tilde{C}}{C^-} \left(\frac{\gamma\mu}{\mu^2 - \gamma^2} \chi_1^-(z) + \frac{\gamma^2}{\mu^2 - \gamma^2} \chi_2^-(z) \right) \quad (100b)$$

and we conclude that the $\chi_{1,2}^-(z)$ solution of the KZ equations corresponds to the second channel of reaction. Using the analytic properties of the hypergeometric functions [19] we obtain :

$$\chi_1^+\left(\frac{1}{z}\right) = \frac{C^+}{C^-} e^{-i\pi\beta} \frac{\gamma\Gamma(1-\mu)\Gamma(-\mu)}{\Gamma(1-\mu-\gamma)\Gamma(1-\mu+\gamma)} \chi_2^-(z) + e^{-i\pi(\beta-\mu)} \frac{\sin \pi\gamma}{\sin \pi\mu} \chi_2^+(z) \quad (101a)$$

$$\chi_2^+\left(\frac{1}{z}\right) = \frac{C^+}{C^-} e^{-i\pi\beta} \frac{\gamma\Gamma(1-\mu)\Gamma(-\mu)}{\Gamma(1-\mu-\gamma)\Gamma(1-\mu+\gamma)} \chi_1^-(z) + e^{-i\pi(\beta-\mu)} \frac{\sin \pi\gamma}{\sin \pi\mu} \chi_1^+(z) \quad (101b)$$

The path of the analytic continuation is such that $-z = e^{i\pi\eta} z$, where $\eta \equiv \eta(\arg \frac{z_1}{z_2}) = 1$.

From (19), (21) one can derive the following connection between the invariant tensors T_i :

$$T_3 = T_1 \frac{\mu}{\sqrt{\mu^2 - \gamma^2}} - T_2 \frac{\gamma}{\sqrt{\mu^2 - \gamma^2}} \quad (102a)$$

$$T_4 = T_2 \frac{\mu}{\sqrt{\mu^2 - \gamma^2}} - T_1 \frac{\gamma}{\sqrt{\mu^2 - \gamma^2}} \quad (102b)$$

Combining these results with the commutation relations (88) :

$$\Psi_2\left(\frac{1}{z}\right) = B_1([m]_n)\Phi_2(z) + B_2([m]_n)\Xi_2(z)$$

$$\text{where } B_1([m]_n) = e^{-i\pi(\beta-\mu)} \frac{\sin \pi\gamma}{\sin \pi\mu}, \quad B_2([m]_n) = e^{-i\pi\beta} \sqrt{1 - \left(\frac{\sin \pi\gamma}{\sin \pi\mu}\right)^2}$$

Therefore the relations between matrix elements in the l. h. s. and the r. h. s.

of (71b) are satisfied provided the constants $C_{[m]_n,j}$ fulfill the relations :

$$\frac{\tilde{C}}{C^+} = \frac{C_{[m]_n+\Delta_n(j),l} \times C_{[m]_n,j}}{C_{[m]_n+\Delta_n(l),j} \times C_{[m]_n,l}} = \quad (103)$$

$$\sqrt{\frac{(\mu + \gamma) \sin \pi(\mu - \gamma) \Gamma(1 - \mu) \Gamma(1 + \mu - \gamma)}{(\mu - \gamma) \sin \pi(\mu + \gamma) \Gamma(1 + \mu) \Gamma(1 - \mu - \gamma)}}$$

The equation (103) does not determine completely the constants $C_{[m]_n,j}$. A solution is :

$$C_{[m]_n,j} = \sqrt{\frac{\prod_{j'} \Gamma(1 + \frac{p_{j'n} - p_{jn}}{k+n})}{\prod_{j'} \Gamma(1 + \frac{p_{j'n} - p_{jn} - 1}{k+n})}} \quad (104)$$

The final form of the level zero matrix elements of $u_{ij}(z)$ is therefore (modulo an overall normalization constant) :

$$\langle (\tilde{m})_n^\infty | u_{ij}(z) | (m)_n^0 \rangle = \frac{C_{[m]_n,j} \langle (\tilde{m})_n | \gamma_{ij} | (m)_n \rangle}{z^{-h(\tilde{m})+h+h(m)}} \quad (105)$$

These matrix elements are well-defined only if $[\tilde{m}]_n$ and $[m]_n$ satisfy the integrability condition (75). It is easy to see that :

$$\lim_{k \rightarrow \infty} \langle (\tilde{m})_n^\infty | u_{ij}(z) | (m)_n^0 \rangle = \langle (\tilde{m})_n | \gamma_{ij} | (m)_n \rangle \quad (106)$$

i. e. in this limit we get the matrix elements of the unit tensor operators. This coincides with the treatment of the "classical limit" of chiral WZW in [15] [3].

Let us consider now the second case for the index structure of the correlation function, when $j = l$:

$$\tilde{\Psi}_2(z_1, z_2) \equiv \langle (\tilde{m})_n^\infty | u_{ij}(z_1) u_{kj}(z_2) | (m)_n^0 \rangle \quad (107)$$

The highest weights of $(\tilde{m})_n^\infty$ and $(m)_n^0$ are connected by :

$$[\tilde{m}]_n = [m]_n + 2\Delta_n(j)$$

The space of invariant couplings $Inv(V_{\tilde{m}}^* \otimes V \otimes V \otimes V_m)$ is one dimensional for the given $|(\tilde{m})\rangle, |(m)\rangle$ and we can write :

$$\tilde{\Psi}_2(z_1, z_2) = \langle (\tilde{m})_n | \gamma_{ij} \gamma_{kj} | (m)_n \rangle \tilde{\psi}(z_1, z_2) \equiv \tilde{T} \tilde{\psi}(z_1, z_2) \quad (108)$$

where γ_{pq} are the unit tensor operators. To simplify the KZ equations we can use the following identities :

$$(t_1 \otimes t_2) \tilde{T} = (1 - \frac{1}{n}) \tilde{T} \quad (109a)$$

$$(t_1 \otimes t_m) \tilde{T} = (\frac{1}{n} - 1 + \frac{1}{2}(C_2([\tilde{m}]_n) - C_2([m]_n + \Delta_n(j)) - C_2)) \tilde{T} \quad (109b)$$

$$(t_2 \otimes t_m) \tilde{T} = \frac{1}{2}(C_2([m]_n + \Delta_n(j)) - C_2([m]_n) - C_2) \tilde{T} \quad (109c)$$

After the substitution $\tilde{\psi}(z_1, z_2) = (z_1 z_2)^{\frac{C_2([\tilde{m}]_n) + C_2([m]_n) - 2C_2}{4(k+n)}} \tilde{\chi}(z)$ we get the following equation :

$$(k+n) \frac{d\tilde{\chi}}{dz} = \left(\frac{2C_2([m]_n + \Delta_n(j)) - C_2([\tilde{m}]_n) - C_2([m]_n)}{4z} + \frac{1-n}{n(1-z)} \right) \tilde{\chi} \quad (110)$$

Using the explicit expression for $C_2([m]_n)$ we obtain the general solution for $\tilde{\chi}(z)$:

$$\tilde{\chi}(z) = C z^{\frac{1-n}{2n(k+n)}} (1-z)^{\frac{n-1}{n(k+n)}} \quad (111)$$

The constant C can be defined from the behavior of the $\tilde{\chi}(z)$ near the point $z = 0$:

$$C = C_{[m]_n + \Delta_n(j), j} \times C_{[m]_n, j} \quad (112)$$

By the analytic continuation $z \rightarrow \frac{1}{z}$ we get :

$$\tilde{\Psi}\left(\frac{1}{z}\right) = e^{i\pi \frac{n-1}{n(k+n)}} \tilde{\Psi}(z) \quad (113)$$

It precisely coincides with the B -matrix exchange relation (72).

This ends the construction. We found the explicit form of the $u_{ij}(z)$ operators in terms of the projected vertex operators [3]. It is given by equation (81) , where the values of the constants $C_{[m]_n, j}$ are calculated above (105).

C. Combining the left and right theories

The situation for the WZW model is completely analogous to the finite dimensional example. The "physical" Hilbert space \mathcal{H}_{phys} is defined by the condition :

$$\forall |\psi\rangle \in \mathcal{H}_{phys} \quad (\vec{p}_l - \vec{p}_r)|\psi\rangle = 0 \quad (114)$$

From this follows that \mathcal{H}_{phys} is :

$$\mathcal{H}_{phys} = \bigoplus_{\{\lambda\}} \mathcal{H}_{\lambda}^l \otimes \mathcal{H}_{\lambda}^r \quad (115)$$

where $\{\lambda\}$ is the set of all integrable representations of $\hat{SU}(n)_k$, the left and right highest weight in any given term being the same. The physical fields, which commute with $\vec{\phi}$ are :

$$g_{ij}(z, \bar{z}) = \sum_k u_{ik}(z) v_{kj}(\bar{z}) \quad (116)$$

It can be shown that in the "physical" Hilbert space :

$$[g_{ij}(z_1, \bar{z}_1), g_{kl}(z_2, \bar{z}_2)] = 0 \quad (117)$$

if $\eta(\arg \frac{z_1}{z_2}) = \eta(\arg \frac{\bar{z}_1}{\bar{z}_2})$.

4. CONCLUSION

In this work we constructed the Hilbert space for the WZW , $\hat{SU}(n)_k$ model in the chiral quantization scheme. The algebra of the chiral group operators can be realized on the space of all integrable representations of $\hat{SU}(n)_k$. The matrix elements of the chiral group elements $u_{ij}(z)$ in the fundamental representation were found explicitly.

We established the connection between these operators and the projected vertex operators which appear in the algebraic treatment of the chiral theory [3]. The chiral group elements are linear combinations with fixed coefficients of the projected vertex operators. Therefore they are not invariant under the "gauge transformations" described in the algebraic treatment [3]. The significance of these particular linear combinations of vertex

operators from the algebraic point of view is still unclear. Our lack of understanding of this feature is closely connected with the missing explanation of the exact action of the quantum group symmetry in the chiral WZW .

We gave the exact rules for combining of the left and right chiral theories, following from the "Gauss law" implied by our gauge choice. The rules reproduce the standard treatment of the WZW model [16] [15].

The generalization of our treatment to the other groups may be a nontrivial task. For the "toy model" a possible difficulty is connected with the generalization of the Gelfand pattern notation, which was crucial in our treatment. For the WZW model the exact form of the commutation relations of the chiral group elements is not known [20] for other groups.

Another open problem is the lagrangian meaning of the solutions obtained combining in a non diagonal way the left and right representations [21].

Acknowledgments

The work of M. L. was supported by Minerva Grant no. 8282 and by the Center for Basic Interactions. We would like to thank A. Schwimmer for many helpful discussions. M. L. is grateful to M. S. Marinov for enlightening remarks about the motion on the group manifold and to L. Feher for clarifying notes about chiral decomposition in WZW model.

Appendix 1

In this Appendix we will prove that the commutation relations of the unit tensor operators

$\left\langle \begin{smallmatrix} i \\ j \end{smallmatrix} \right\rangle$ are the same as the commutation relations of the u_{ji} operators.

Let us first consider the $U(n)$ tensor operators. Their definition coincides with the definition of the $SU(n)$ tensor operators, but in this case m_{nn} in the Gelfand pattern may be different from zero. The proof can be given by induction. For $U(2)$ we have only four unit tensor operators (next to each operator we indicate its action on a generic state $|(m)\rangle$):

$$1) \quad \left\langle \begin{smallmatrix} 1 \\ 10 \\ 1 \end{smallmatrix} \right\rangle \equiv \left\langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right\rangle \quad ; \quad \left\langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} m_{12} & m_{22} \\ m_{11} \end{smallmatrix} \right\rangle =$$

$$\sqrt{\frac{m_{11} - m_{22} + 1}{m_{12} - m_{22} + 1}} \left| \begin{array}{cc} m_{12} + 1 & m_{22} \\ m_{11} + 1 & \end{array} \right\rangle$$

$$2) \quad \left\langle \begin{array}{c} 1 \\ 10 \\ 0 \end{array} \right\rangle \equiv \left\langle \begin{array}{c} 1 \\ 2 \end{array} \right\rangle \quad ; \quad \left\langle \begin{array}{c} 1 \\ 2 \end{array} \right\rangle \left| \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right\rangle =$$

$$-\sqrt{\frac{m_{12} - m_{11} + 1}{m_{12} - m_{22} + 1}} \left| \begin{array}{cc} m_{12} + 1 & m_{22} \\ m_{11} & \end{array} \right\rangle$$

$$3) \quad \left\langle \begin{array}{c} 0 \\ 10 \\ 1 \end{array} \right\rangle \equiv \left\langle \begin{array}{c} 2 \\ 1 \end{array} \right\rangle \quad ; \quad \left\langle \begin{array}{c} 2 \\ 1 \end{array} \right\rangle \left| \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right\rangle =$$

$$\sqrt{\frac{m_{12} - m_{11}}{m_{12} - m_{22} + 1}} \left| \begin{array}{cc} m_{12} & m_{22} + 1 \\ m_{11} + 1 & \end{array} \right\rangle$$

$$4) \quad \left\langle \begin{array}{c} 0 \\ 10 \\ 0 \end{array} \right\rangle \equiv \left\langle \begin{array}{c} 2 \\ 2 \end{array} \right\rangle \quad ; \quad \left\langle \begin{array}{c} 2 \\ 2 \end{array} \right\rangle \left| \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right\rangle =$$

$$\sqrt{\frac{m_{11} - m_{22}}{m_{12} - m_{22} + 1}} \left| \begin{array}{cc} m_{12} & m_{22} + 1 \\ m_{11} & \end{array} \right\rangle$$

By direct calculation it is possible to obtain (for $i \neq k$) :

$$\left\langle \begin{array}{c} i \\ j \end{array} \right\rangle \left\langle \begin{array}{c} k \\ l \end{array} \right\rangle = \left\langle \begin{array}{c} i \\ l \end{array} \right\rangle \left\langle \begin{array}{c} k \\ j \end{array} \right\rangle \frac{\hbar}{p_i - p_k} + \left\langle \begin{array}{c} k \\ l \end{array} \right\rangle \left\langle \begin{array}{c} i \\ j \end{array} \right\rangle \sqrt{1 - \frac{\hbar^2}{(p_i - p_k)^2}} \quad (118)$$

where

$$p_i \left| \begin{array}{c} [m]_n \\ (m)_{n-1} \end{array} \right\rangle = \hbar(m_{in} + n - i) \left| \begin{array}{c} [m]_n \\ (m)_{n-1} \end{array} \right\rangle \equiv \hbar p_{in} \left| \begin{array}{c} [m]_n \\ (m)_{n-1} \end{array} \right\rangle \quad (119)$$

Let us assume that the statement is true for the $U(n-1)$ unit tensor operators. The product of two unit tensor operators acts on the generic state $| (m) \rangle$ as it follows from (40) :

$$\left\langle \begin{array}{c} i \\ j \end{array} \right\rangle_n \left\langle \begin{array}{c} k \\ l \end{array} \right\rangle_n \left| \begin{array}{c} [m]_n \\ (m)_{n-1} \end{array} \right\rangle = \sum_{rs} \left[\begin{array}{cc} i & [m]_n + \Delta_n(k) \\ r & [m]_{n-1} + \Delta_{n-1}(s) \end{array} \right] \left[\begin{array}{cc} k & [m]_n \\ s & [m]_{n-1} \end{array} \right] \left\langle \begin{array}{c} r \\ t \end{array} \right\rangle_{n-1} \left\langle \begin{array}{c} s \\ q \end{array} \right\rangle_{n-1} \times \quad (120)$$

$$\times \left| \begin{array}{c} [m]_n + \Delta_n(i) + \Delta_n(k) \\ (m)_{n-1} \end{array} \right\rangle$$

where the subscript n , $n-1$ designates that the relevant unit tensor operators correspond to $U(n)$, $U(n-1)$. Equation (46) gives :

$$\left[\begin{array}{cc} k & [m]_n \\ s & [m]_{n-1} \end{array} \right] = \sqrt{\frac{\prod_{s' \neq s}^{n-1} (p_{kn} - p_{s',n-1}) \prod_{k' \neq k}^n (p_{s,n-1} - p_{k'n} + 1)}{\prod_{k' \neq k}^n (p_{kn} - p_{k'n}) \prod_{s' \neq s}^{n-1} (p_{s,n-1} - p_{s',n-1} + 1)}} \quad (121)$$

$$\left[\begin{array}{cc} i & [m]_n + \Delta(k)_n \\ r & [m]_{n-1} + \Delta(s)_{n-1} \end{array} \right] = \sqrt{\frac{\prod_{r' \neq r}^{n-1} (p_{in} - p_{r',n-1} - \delta_{r's}) \prod_{i' \neq i}^n (p_{r,n-1} - p_{i'n} - \delta_{i'k} + 1)}{\prod_{i' \neq i}^n (p_{in} - p_{i'n} - \delta_{i'k}) \prod_{r' \neq r}^{n-1} (p_{r,n-1} - p_{r',n-1} - \delta_{r's} + 1)}} \quad (122)$$

where $p_{ij} = m_{ij} + j - i$. By straightforward calculation it can be shown, that :

$$\left[\begin{array}{cc} i & [m]_n + \Delta(k)_n \\ r & [m]_{n-1} + \Delta(s)_{n-1} \end{array} \right] \left[\begin{array}{cc} k & [m]_n \\ s & [m]_{n-1} \end{array} \right] = C_1 \left[\begin{array}{cc} i & [m]_n + \Delta(k)_n \\ s & [m]_{n-1} + \Delta(r)_{n-1} \end{array} \right] \left[\begin{array}{cc} k & [m]_n \\ r & [m]_{n-1} \end{array} \right] + \quad (123)$$

$$C_2 \left[\begin{array}{cc} k & [m]_n + \Delta(i)_n \\ s & [m]_{n-1} + \Delta(r)_{n-1} \end{array} \right] \left[\begin{array}{cc} i & [m]_n \\ r & [m]_{n-1} \end{array} \right]$$

where

$$C_1 = \frac{p_{in} - p_{kn} + p_{r,n-1} - p_{s,n-1}}{(p_{in} - p_{kn}) \sqrt{(p_{r,n-1} - p_{s,n-1} + 1)(p_{r,n-1} - p_{s,n-1} - 1)}} \quad (124a)$$

$$C_2 = \frac{p_{r,n-1} - p_{s,n-1}}{p_{in} - p_{kn}} \sqrt{\frac{(p_{in} - p_{kn} + 1)(p_{in} - p_{kn} - 1)}{(p_{r,n-1} - p_{s,n-1} + 1)(p_{r,n-1} - p_{s,n-1} - 1)}} \quad (124b)$$

if $r \neq s$ and

$$C_1 = \frac{1}{p_{in} - p_{kn}} \quad (125a)$$

$$C_2 = \frac{\sqrt{(p_{in} - p_{kn} + 1)(p_{in} - p_{kn} - 1)}}{p_{in} - p_{kn}} \quad (125b)$$

if $r = s$. By our assumption about $U(n-1)$ unit tensor operators :

$$\left\langle \begin{matrix} r \\ t \end{matrix} \right\rangle_{n-1} \left\langle \begin{matrix} s \\ q \end{matrix} \right\rangle_{n-1} = \left\langle \begin{matrix} r \\ q \end{matrix} \right\rangle_{n-1} \left\langle \begin{matrix} s \\ t \end{matrix} \right\rangle_{n-1} \frac{1}{p_{r,n-1} - p_{s,n-1}} + \quad (126)$$

$$\left\langle \begin{matrix} s \\ q \end{matrix} \right\rangle_{n-1} \left\langle \begin{matrix} r \\ t \end{matrix} \right\rangle_{n-1} \sqrt{1 - \frac{1}{(p_{r,n-1} - p_{s,n-1})^2}}$$

if $r \neq s$ and

$$\left\langle \begin{matrix} r \\ t \end{matrix} \right\rangle_{n-1} \left\langle \begin{matrix} r \\ q \end{matrix} \right\rangle_{n-1} = \left\langle \begin{matrix} r \\ q \end{matrix} \right\rangle_{n-1} \left\langle \begin{matrix} r \\ t \end{matrix} \right\rangle_{n-1} \quad (127)$$

By substitution of (124), (127) into (120) we obtain the desired result. The proof in the case, when $i = k$ or $j(l) = n$ is also straightforward and completely analogous to the previous one. For the $SU(n)$ unit tensor operators we will obtain formally the same result, if we will introduce the operator p_n ; $p_n \equiv 0$, because the only difference between an $U(n)$ irrep $[m]_n$ and the corresponding $SU(n)$ irrep $[m]_n$ is the fact that $m_{nn} = 0$ for the $SU(n)$ irrep.

Appendix 2

Proof of relations (88) , (109)

Let t_{pq}^a denote the matrix elements of the $SU(n)$ generators in the fundamental representation with normalization chosen as in the section 2 C and $t_{(m)(m')}^a$ in the irrep $[m]_n$. Let $\langle (\tilde{m})_n | \gamma_{ij} \gamma_{kl} | (m)_n \rangle$ denote the matrix elements of the product of two unit tensor operators.

Then :

$$\sum_a \sum_{pq} t_{ip}^a t_{kq}^a \langle (\tilde{m}) | \gamma_{pj} \gamma_{ql} | (m) \rangle = \quad (128)$$

$$\langle (\tilde{m}) | \gamma_{kj} \gamma_{il} | (m) \rangle - \frac{1}{n} \langle (\tilde{m}) | \gamma_{ij} \gamma_{kl} | (m) \rangle$$

which proves (88a). Notice also that from (19) it follows that :

$$\langle (\tilde{m}) | \gamma_{kj} \gamma_{il} | (m) \rangle = \langle (\tilde{m}) | \gamma_{ij} \gamma_{kl} | (m) \rangle \quad (129)$$

which proves (109a). We will prove now (88b), (88c).

$$\sum_a \sum_{p, (m)'} t_{ip}^a t_{kq}^a \langle (\tilde{m}) | \gamma_{pj} \gamma_{kl} | (m') \rangle = \quad (130)$$

$$\sum_a \langle (\tilde{m}) | [J^a, \gamma_{ij}] \gamma_{kl} J^a | (m) \rangle =$$

$$\begin{aligned} & \sum_a (-\langle (\tilde{m}) | [J^a, \gamma_{ij}] [J^a, \gamma_{kl}] | (m) \rangle + \langle (\tilde{m}) | [J^a, \gamma_{ij}] J^a \gamma_{kl} | (m) \rangle) = \\ & - \sum_a \sum_{pq} t_{ip}^a t_{kq}^a \langle (\tilde{m}) | \gamma_{pj} \gamma_{ql} | (m) \rangle + \frac{1}{2} \langle (\tilde{m}) | [\sum_a J^a J^a, \gamma_{ij}] \gamma_{kl} | (m) \rangle - \\ & \langle (\tilde{m}) | \sum_a [J^a, [J^a \gamma_{ij}]] \gamma_{kl} | (m) \rangle = -\langle (\tilde{m}) | \gamma_{kj} \gamma_{il} | (m) \rangle + \end{aligned}$$

$$\frac{1}{2} \left(\frac{2}{n} + C_2([\tilde{m}]_n) - C_2([m]_n + \Delta_n(l)) - C_2 \right) \langle (\tilde{m}) | \gamma_{ij} \gamma_{kl} | (m) \rangle$$

To show (109b) we must take into account (129).

$$\sum_a \sum_{q, (m)'} t_{kq}^a t_{ip}^a \langle (\tilde{m}) | \gamma_{ij} \gamma_{ql} | (m') \rangle = \quad (131)$$

$$\sum_a \langle (\tilde{m}) | \gamma_{ij} [J^a, \gamma_{kl}] J^a | (m) \rangle =$$

$$\frac{1}{2} \langle (\tilde{m}) | \gamma_{ij} [\sum_a J^a J^a, \gamma_{kl}] | (m) \rangle - \langle (\tilde{m}) | \gamma_{ij} \sum_a [J^a, [J^a \gamma_{kl}]] | (m) \rangle =$$

$$\frac{1}{2}(C_2([m]_n + \Delta_n(l)) - C_2([m]_n) - C_2)\langle(\tilde{m})|\gamma_{ij}\gamma_{kl}|(m)\rangle$$

which proves (88d), (88e).

Appendix 3

Proof of the formulas (100).

We use the following identities of the hypergeometric functions [19] :

$$(\gamma - \alpha - 1)F(\alpha, \beta, \gamma, z) + \alpha F(\alpha + 1, \beta, \gamma, z) - \quad (132a)$$

$$(\gamma - 1)F(\alpha, \beta, \gamma - 1, z) = 0$$

$$(\gamma - \beta - 1)F(\alpha, \beta, \gamma, z) + \beta F(\alpha, \beta + 1, \gamma, z) - \quad (132b)$$

$$(\gamma - 1)F(\alpha, \beta, \gamma - 1, z) = 0$$

From (132) it follows that :

$$F(\alpha, \beta, \gamma, z) = \frac{\alpha(\gamma - \beta)}{(\beta - 1)(\gamma - \alpha - 1)} F(\alpha + 1, \beta - 1, \gamma, z) + \quad (133)$$

$$\frac{(\gamma - 1)(\beta - \alpha)}{(\beta - 1)(\gamma - \alpha - 1)} F(\alpha, \beta - 1, \gamma - 1, z)$$

and by substitution of the explicit expression for $\xi_1(z)$ and $\chi_{1,2}^-(z)$ we will obtain (100a).

From the :

$$(\gamma - \alpha - 1)F(\alpha, \beta, \gamma, z) + \quad (134a)$$

$$\alpha F(\alpha + 1, \beta, \gamma, z) - (\gamma - 1)F(\alpha, \beta, \gamma - 1, z) = 0$$

$$\beta z F(\alpha, \beta + 1, \gamma + 1, z) = \gamma F(\alpha, \beta, \gamma, z) - \gamma F(\alpha - 1, \beta, \gamma, z) \quad (134b)$$

it follows that :

$$z F(\alpha, \beta + 1, \gamma + 1, z) = \quad (135)$$

$$\frac{\gamma(1 - \gamma)}{\beta(\alpha - \gamma)} (F(\alpha, \beta, \gamma, z) - F(\alpha - 1, \beta, \gamma - 1, z))$$

which proves (100b).

REFERENCES

- [1] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333
- [2] A. Tsuchiya and Y. Kanie, Lett. Math. Phys. **13** (1987) 303
- [3] G. Moore, N. Seiberg, Commun. Math. Phys. **123** (1989) 177; Phys. Lett. **B212** (1988) 451; Proceeding of the Trieste Spring School (1989)
- [4] E. Witten Commun. Math. Phys. **92** (1984) 455
- [5] B. Blok, Phys. Lett. **B233** (1989)359
- [6] L. D. Faddeev, Commun. Math. Phys. **132** (1990)131
- [7] M. Chu, P. Goddard, I. Halliday, D. Olive, A. Schwimmer, Phys. Lett. **B266** (1991)71
- [8] M. Chu, P. Goddard, Nucl. Phys. **B445** (1995)145
- [9] M. S. Marinov and M. V. Terentyev, Fortschritte der Physik **27** (1979)511
- [10] A. Yu. Alekseev, L. D. Faddeev, Commun. Math. Phys. **141** (1991)413
- [11] M. Chu, P. Goddard, Phys. Lett. **B337** (1994)285
- [12] D. P. Zelobienko, "Compact Lie Groups and their Representations"
- [13] L. C. Biedenharn and J. D. Louck, Commun. Math. Phys. **8** (1968)80
L. C. Biedenharn and J. D. Louck, J. Math. Phys. **11** (1970)2368
- [14] A. Schwimmer, unpublished
- [15] D. Gepner and E. Witten, Nucl. Phys. **B278** (1986)493
- [16] V. G. Knizhnik and A. B. Zamoldchikov, Nucl. Phys. **B247** (1984)83
- [17] I. B. Frenkel, N. Yu. Reshetikhin, Commun. Math. Phys. **146** (1992)1
- [18] D. Gepner, J. Fuchs, Nucl. Phys. **B413** (1994)614

- [19] Z. X. Wang and D. R. Guo "Special functions"
- [20] J. Balog, L. Dabrowski and L. Feher, Phys. Lett. **B244** (1990) 227
- [21] G. Felder, K. Gawedzki and A. Kupiainen, Commun. Math. Phys. **130** (1990)